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A new exact approach for determining natural frequencies and mode shapes of non-uniform shear beams with arbitrary distribution of mass or stiffness

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Abstract

A new exact approach which combines the basic solutions with unit matrix property and recurrence formula for determining natural frequencies and mode shapes of non-uniform shear beams is presented in this paper. The function for describing the distribution of mass is arbitrary, and the distribution of shear stiffness is expressed as a functional relation with the mass distribution and vice versa. The governing equation for free vibration of a non-uniform shear beam is reduced to a differential equation of the second-order without the first-order derivative by means of functional transformation. Then, this kind of differential equation is reduced to Bessel equations and other solvable equations for six cases. The exact solutions of mode shape functions are thus found. The basic solutions, which have a unit matrix property, are derived and used to obtain the frequency equations and mode shapes of multi-step shear beams with varying cross-sections. Numerical examples show that the calculated natural frequencies and mode shapes of two symmetric buildings are very close to the corresponding field measured data, suggesting that the proposed methods are applicable to engineering application and practice. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Non-uniform beam; Natural frequencies; Mode shapes; Bessel functions

1. Introduction

It has been recognized that the lateral deflection of most buildings is not purely flexural, but there is a considerable contribution from shear deflections in most cases. If shear deformation is dominated in the total deformation of buildings in their horizontal vibrations, such structures are usually called shear-type buildings. The field measured data (e.g., Korqingskee, 1953; Wang, 1958; Ishizaki and Hatakeynan,

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1964; Li et al., 1994; Jeary, 1997) have shown that the shear deformation is usually dominant in the total deformation of frame buildings and multi-story brick buildings in their horizontal vibrations. Korqingskee (1953) investigated free vibration of frame buildings which were considered as a multi-step cantilever shear beam and in which, each step of the beam has constant mass and stiffness. Wang (1958) suggested that frame buildings can be simplified as a one-step shear beam with continuously varying cross-section for free vibration analysis. He assumed that the mass of the shear beam is proportional to its stiffness. However, this assumption is not suitable for many multi-story buildings. This is due to the fact that the mass of floors is a major part of the total mass of a multi-story building, and the variation of mass at different floors is not significant. Thus, the distribution of mass with height is not necessarily proportional to that of the stiffness. Wang (1978) simplified frame buildings as a special shear beam with uniform mass, but variable stiffness in free vibration analysis. He derived analytical solutions for such problems. Wang (1978) and Cao et al. (1992) investigated free vibration of a one-story industrial building. They considered such a building as a uniform shear beam with rectilinear springs representing the elastic supports of the closely spaced columns to the roof system of the building. Li et al. (1994, 1996, 1998) proposed a general approach for determining natural frequencies and mode shapes of multi-story buildings which were treated as a one-step or a multi-step cantilever shear beam, or a spring-free shear beam (Li et al., 1994), with continuously varying cross-section, and the mass distribution of such a beam, in general, is not proportional to that of the shear stiffness. The expressions for describing the distributions of mass and stiffness were selected as power functions or exponential functions. The exact solutions were obtained. A review of technical literature dealing with this problem indicates that generally the authors of the previous studies have directed their investigations to special functions for describing the distributions of mass per unit length and shear stiffness in order to derive closed-form solutions.

Panayatounakos (1994) obtained classes of analytical solutions for the linear ordinary differential equation of variable coefficients governing the stability analysis of bars with varying cross-section. However, exact analytical solutions for free vibration of non-uniform shear beams with arbitrary distribution of mass or stiffness have not been obtained in the literature. The objective of this paper is to present a new exact approach for determining the natural frequencies and mode shapes of one-step and multi-step non-uniform shear beams with concentrated masses and rectilinear springs. The exact approach combines the basic solutions with unit matrix property and recurrence formulae developed in this paper. The mass distribution of a non-uniform shear beam considered in this paper is arbitrary, and the stiffness distribution is a functional expression of the mass distribution and vice versa. Thus, classes of useful solutions in engineering practices are obtained. The numerical examples presented in this paper show that the proposed procedure is an exact method, and with the proposed procedure, there is no need to take any matrix multiplication for free vibration analysis of a multi-step non-uniform beam with many translational springs.

Apart from the several analytical methods for analyzing limited classes of non-uniform beams or bars, many approximate methods have been developed. These include the Ritz method, the finite strip method (FSM) and the finite element method (FEM). In general, the Ritz method can provide accurate solutions, however, it depends on the choice of global admissible functions. Liew et al. (1997) and (1998) have developed efficient three-dimensional Ritz algorithms for the free vibration analysis of elastic solid cylinders. Their method that was developed based on a global three-dimensional elasticity energy principle with polynomial-based displacement shape functions is capable of extracting all possible modes of vibration for elastic solid cylinders. Their work provided useful benchmarking reference for research development in simplified beam theories because three-dimensional analysis is an important base for exact comparison studies. The FEM and FSE have been developed and widely applied to vibration analysis of various non-uniform structural members over the years. Compared with FEM, the main advantage of FSE is its efficiency, in particular, for structural members with regular geometry.

It is necessary to point out that only symmetric buildings may be reasonably simplified as shear beams for dynamic analysis. As pointed by Wilkinson and Thambiratnam (1995), asymmetric buildings are torsionally unbalanced and display complex interaction between the lateral and torsional components of their response. Numerical methods are widely used in dynamic analysis of asymmetric buildings. Extensive work on the vibration response of asymmetric buildings has been carried out by Thambiratnam and Irvine (1989), Wilkinson and Thambiratnam (1995), and simple three-dimensional computer models based on energy methods that are useful to practicing engineers for design purposes, have been developed by them.

In fact, the free vibration of non-uniform beams with arbitrary distribution of mass or stiffness can be analyzed using numerical methods (e.g., FEM). However, the present exact solutions that can be easily implemented could provide adequate insight into the physics of the problem and supplement the existing database, and further serve as the benchmark for researchers and engineers to examine the merits of new numerical method and development in this field. Therefore, it is always desirable to obtain exact solutions to such problems.

2. Solutions for one-step non-uniform shear beams

The governing differential equation for mode shapes of a one-step shear beam can be written as (Li et al., 1994)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[K(x)\frac{\mathrm{d}X(x)}{\mathrm{d}x}\right]+\bar{m}(x)\omega^2X(x)=0 \tag{1}$$

in which $X(x)$, $K(x)$, $\bar{m}(x)$ and ω are the mode shape function, shear stiffness, mass per unit length and circular natural frequency of the shear beam, respectively.

In order to solve Eq. (1) for an arbitrary distribution of shear stiffness or mass, it is assumed that

$$K(x)=\text{arbitrary},\quad \bar{m}(x)=K^{-1}(x)p(r),\quad r=\int K^{-1}(x)\mathrm{d}x,\quad X(x)=X(r) \tag{2}$$

or

$$\bar{m}(x)=\text{arbitrary},\quad K(x)=\bar{m}^{-1}(x)p(r),\quad r=\int K^{-1}(x)\mathrm{d}x,\quad X(x)=X(r) \tag{3}$$

Substituting Eq. (2) or (3) into Eq. (1), we have

$$\frac{\mathrm{d}^2X(r)}{\mathrm{d}t^2}+\omega^2p(r)X(r)=0 \tag{4}$$

It is noted that r is a function of x, and $p(r)$ is a functional expression. Thus, the solution of Eq. (4) represents a class of solution. On the other hand, it is easier to find the exact solution of Eq. (4) than solving Eq. (1). It is decided to derive the solution of Eq. (4). However, the solutions of $X(r)$ are dependent on the expression of $p(r)$. Several important cases are considered and discussed as follows.

Case 1.

$$p(r)=a\,\mathrm{e}^{br}-c,\quad c>0 \tag{5}$$

where a, b, c are parameters which can be determined by the values of $\bar{m}(x)$ and $K(x)$ according to the relations given in Eq. (2) or (3).

Substituting Eq. (5) into Eq. (4) gives that

$$\frac{d^2X(r)}{dr^2} + \omega^2\left(a\,e^{br} - c\right)X(r) = 0 \tag{6}$$

The above equation can be reduced to a Bessel equation by the substitutions

$$\zeta = e^{br/2} \tag{7}$$

as follows

$$\frac{d^2X}{d\zeta^2} + \frac{1}{\zeta}\frac{dX}{d\zeta} + \left(\alpha^2 - \frac{\nu^2}{\zeta^2}\right)X = 0 \tag{8}$$

in which

$$\alpha = \frac{2\omega a^{1/2}}{|b|}, \qquad \nu = \frac{2\omega c^{1/2}}{|b|} \tag{9}$$

$|b|$ represents the absolute value of b.

The general solution of Eq. (8) can be written as

$$X = \begin{cases} c_1 J_\nu(\alpha\,e^{br/2}) + c_2 J_{-\nu}(\alpha\,e^{br/2}), & \nu = \text{a non-integer} \\ c_1 J_\nu(\alpha\,e^{br/2}) + c_2 Y_\nu(\alpha\,e^{br/2}), & \nu = \text{an integer} \end{cases} \tag{10}$$

where c_1 and c_2 are constants of integration, which can be determined according to the boundary conditions of the shear beam. J_ν and Y_ν are Bessel functions of the first, second kind of order ν, respectively.

If $c = 0$, then

$$X = c_1 J_0(\alpha\,e^{br/2}) + c_2 Y_0(\alpha\,e^{br/2}) \tag{11}$$

If $b = c = 0$, then $p(r) = a$, which corresponds to a uniform shear beam. The general solution for mode shapes of a uniform shear beam can be expressed as

$$X = c_1 \sin\left(\frac{\sqrt{\bar{m}}}{K}\omega x\right) + c_2 \cos\left(\frac{\sqrt{\bar{m}}}{K}\omega x\right) \tag{12}$$

where $\bar{m}$ and K are the mass per unit length and shear stiffness, respectively.

Case 2.

$$p(r) = (a + br)^c \tag{13}$$

where a, b, c are constants which can be determined by the values of $\bar{m}(x)$ and $K(x)$ at control sections.

Substituting Eq. (13) into Eq. (4) we have

$$\frac{d^2X(r)}{dr^2} + \omega^2(a + br)^c X(r) = 0 \tag{14}$$

Using the following functional transformation

$$X = \zeta^{\nu} Z, \qquad \zeta = (a + br)^{1/2\nu} \tag{15}$$

Eq. (14) is reduced to a Bessel equation of order ν as follows:

$$\frac{d^2 Z}{d\zeta^2} + \frac{1}{\zeta}\frac{dZ}{d\zeta} + \left(\bar{\alpha}^2 - \frac{\nu^2}{\zeta^2}\right) Z = 0 \tag{16}$$

in which

$$\bar{\alpha} = \frac{2\nu\omega}{|b|}, \qquad \nu = \frac{1}{c+2} \tag{17}$$

The mode shape function can be expressed as

$$X = \begin{cases} (a+br)^{1/2}\left\{c_1 J_\nu\left[\bar{\alpha}(a+br)^{1/2}\right] + c_2 J_{-\nu}\left[\bar{\alpha}(a+br)^{1/2}\right]\right\}, & \nu = \text{a non-integer} \\ (a+br)^{1/2}\left\{c_1 J_\nu\left[\bar{\alpha}(a+br)^{1/2}\right] + c_2 Y_\nu\left[\bar{\alpha}(a+br)^{1/2}\right]\right\}, & \nu = \text{an integer} \end{cases} \tag{18}$$

If $c = -2$, then Eq. (14) becomes an Euler equation, the general solution of which can be written as

$$X = \begin{cases} (a+br)^{1/2}\left\{c_1 \sin\left[\tilde{\alpha}\ln(a+br)\right] + c_2 \cos\left[\tilde{\alpha}\ln(a+br)\right]\right\} & \text{for } 4\omega^2 - b^2 > 0 \\ c_1(a+br)^{1/2+\tilde{\alpha}} + c_2(a+br)^{1/2-\tilde{\alpha}} & \text{for } 4\omega^2 - b^2 < 0 \\ c_1(a+br)^{1/2} + c_2(a+br)^{1/2}\ln(a+br) & \text{for } 4\omega^2 - b^2 = 0 \end{cases} \tag{19}$$

where

$$\tilde{\alpha} = \frac{|4\omega^2 - b^2|^{1/2}}{2|b|} \tag{20}$$

Case 3.

$$P(r) = a(1+br)^c \tag{21}$$

This case is an alteration of Case 2. The mode shape function can be written as

$$X = \begin{cases} (1+br)^{1/2}\left\{c_1 J_\nu\left[\lambda(1+br)^{1/2\nu}\right] + c_2 J_{-\nu}\left[\lambda(1+br)^{1/2\nu}\right]\right\} & \nu = \text{a non-integer} \\ (1+br)^{1/2}\left\{c_1 J_\nu\left[\lambda(1+br)^{1/2\nu}\right] + c_2 Y_\nu\left[\lambda(1+br)^{1/2\nu}\right]\right\} & \nu = \text{an integer} \end{cases} \tag{22}$$

where

$$\lambda = \frac{2\omega\nu a^{1/2}}{|b|}$$

If $c = -2, \nu = \infty$, then, the solutions, Eq. (22), are not valid for this case. The expression of mode shape function for this case is similar to that given in Eq. (19).

Case 4.

$$P(r) = a\varepsilon^2 \left(r^2 + b\right)^{-2} \quad a > 0, b > 0 \tag{23}$$

The mode shape function for this case is given by

$$X = \left(r^2 + b\right)^{1/2}(c_1 \sin \xi + c_2 \cos \xi) \tag{24}$$

where

$$\xi = \left(\frac{a\varepsilon^2 + b}{b}\right)^{1/2} \arctan\frac{r}{b^{1/2}} \tag{25}$$

Case 5.

$$P(r) = a\varepsilon^2 (r^2 - b)^{-2} \quad a > 0, b > 0 \tag{26}$$

The mode shape function for this case is given by

$$X = (b - r^2)^{1/2}(c_1 \sin \xi + c_2 \cos \xi) \tag{27}$$

where

$$\xi = \frac{1}{2}\left(\frac{a\varepsilon^2 - b}{b}\right)^{1/2} \ln\frac{b^{1/2} + r}{b^{1/2} - r} \tag{28}$$

Case 6.

$$p(r) = -c\varepsilon^2 \left[r^2 - (a + b)r + ab\right]^{-2} \tag{29}$$

The mode shape function for this case is as

$$X = |r - a|^{\frac{1+\gamma}{2}} \left(c_1 |r - b|^{\frac{1-\gamma}{2}} + c_2 |r - b|^{\frac{1+\gamma}{2}}\right) \tag{30}$$

where

$$\gamma^2 = \frac{4c\varepsilon^2}{(a - b)^2} + 1 \neq 0, \quad a \neq b$$

When we select the expression of $p(r)$, we should consider that not only the solution of Eq. (4) can be expressed in a closed-form, but also the actual distributions of shear stiffness and mass can be exactly or approximately described by $p(r)$.

In order to establish the frequency equations for various boundary conditions, it will be convenient to express all the above solutions as a unified form

$$X = c_1 S_1(x) + c_2 S_2(x) \tag{31}$$

where $S_1(x)$ and $S_2(x)$ are special solutions of mode shape functions which can be determined from Eqs. (10)–(30).

Based on the linearly independent special solutions, $S_1(x)$ and $S_2(x)$, in order to simplify the analysis, the linearly independent basic solutions, $\bar{S}_1(x)$ and $\bar{S}_2(x)$, which have a unit matrix property at the origin of co-ordinate system

$$\begin{bmatrix} \bar{S}_1(0) & \bar{S}_1'(0) \\ \bar{S}_2(0) & \bar{S}_2'(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{32}$$

can be easily constructed by

$$\begin{bmatrix} \bar{S}_1(x) \\ \bar{S}_2(x) \end{bmatrix} = \begin{bmatrix} S_1(0) & S_1'(0) \\ S_2(0) & S_2'(0) \end{bmatrix}^{-1} \begin{bmatrix} S_1(x) \\ S_2(x) \end{bmatrix} \tag{33}$$

where

$$\bar{S}_i'(0) = \frac{\mathrm{d}S_i(x)}{\mathrm{d}x}|_{x=0} \quad i = 1, 2 \tag{34}$$

In general, the mode shape function of a one-step beam can be expressed in terms of the basic solutions as follows

$$X(x) = X_0\bar{S}_1(x) + \frac{Q_0}{K(0)}\bar{S}_2(x) \tag{35}$$

where $X_0 = X(0)$ and $Q_0 = Q(0)$ are the displacement and shear force of the shear beam at $x = 0$, respectively.

The frequency equation can be easily established by using the basic solutions of mode shape functions and the boundary conditions of the beam as follows

1. *Fixed-free beam*. If a shear beam is fixed at the left end, then $X_0 = 0$, the mode shape function, Eq. (35), becomes

$$X(x) = \frac{Q_0}{K(0)}\bar{S}_2(x) \tag{36}$$

If the right end of this shear beam is free, then $X'(L) = 0$, the frequency equation is

$$\bar{S}_2'(L) = 0 \tag{37}$$

2. *Fixed-fixed beam*. The mode shape function of a fixed-fixed beam has the same expression as that given in Eq. (36), but the frequency equation is

$$\bar{S}_2(L) = 0 \tag{38}$$

3. *Free-free beam*. Because the shear force at the free end is equal to zero, $Q_0 = 0$, the mode shape function, Eq. (35), becomes

$$X(x) = X_0\bar{S}_1(x) \tag{39}$$

The frequency equation can be established according to the boundary condition of the right end, i.e., the shear force is equal to zero, as follows

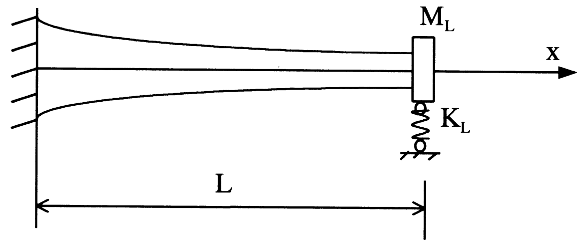

Fig. 1. A fixed-spring beam with a concentrated mass.

$$\bar{S}_1'(L) = 0 \tag{40}$$

4. *Fixed-spring beam* with a concentrated mass at the spring end (Fig. 1).
   The boundary conditions for this case are

$$x = 0,\ X(0) = 0,\ x = L,\ X'(L) = \frac{X(L)}{K(L)}\left(M_L\omega^2 - K_L\right) \tag{41}$$

where $K(L)$ and K_L are the shear stiffness of the beam at $x = L$ and the spring stiffness at the right end, respectively.

The mode shape function has the same expression as that given in Eq. (36), but the frequency equation is

$$K(L)\bar{S}_2'(L) = \bar{S}_2(L)\left(M_L\omega^2 - K_L\right) \tag{42}$$

If the right end is free, but with a concentrated mass, then we should let $K_L = 0$ in the above equation.

5. *Spring-spring beam* with concentrated masses at the ends (Fig. 2).
   The boundary conditions for this case are

$$x = 0, \qquad K(0)X'(0) = -X_0\left(M_0\omega^2 - K_0\right)$$

$$x = L, \qquad K(L)X'(L) = X_L\left(M_L\omega^2 - K_L\right) \tag{43}$$

where $K(0)$ and K_0 are the shear stiffness of the beam at $x = 0$ and the spring stiffness at the left end, respectively.

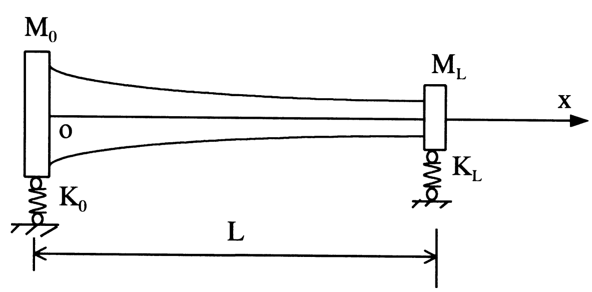

Fig. 2. A spring-spring beam with concentrated masses at the ends.

The mode shape function can be written as

$$X(x) = X_0\left[\bar{S}_1(x) - \frac{\bar{S}_2(x)}{K(0)}\left(M_0\omega^2 - K_0\right)\right] \tag{44}$$

Using the boundary condition at $x = L$ and setting $X_0 = 1$, one obtains the frequency equation as follows

$$K(L)\bar{S}_1'(L) - \frac{K(L)}{K(0)}\left(M_0\omega^2 - K_0\right)\bar{S}_2'(L) = \left(M_L\omega^2 - K_L\right)\left[\bar{S}_1(L) - \frac{\bar{S}_2(L)}{K(0)}\left(M_0\omega^2 - K_0\right)\right] \tag{45}$$

6. *Spring-spring beam* with concentrated masses and rectilinear springs at the ends and the $(n-1)$ intermediate points (Fig. 3).

The boundary conditions for this case are the same as those described in Eq. (43) and the mode shape function can be written as

$$X(x) = X_1(x) - \sum_{i=1}^{n}\frac{X_i(l_i)}{K(l_i)}\left(M_i\omega^2 - K_i\right)\bar{S}_2(x - l_i)H(x - l_i) \tag{46}$$

in which $X_1(x)$ is the mode shape function of the first segment and $H(\cdot)$ is Heaviside function.

According to the boundary condition at $x = 0$, we have

$$X_1(x) = X_0\left[\bar{S}_1(x) - \frac{\bar{S}_2(x)}{K(0)}\left(M_0\omega^2 - K_0\right)\right] \tag{47}$$

For the ith segment we have the recurrence formula as

$$X_i(x) = X_{i-1}(x) - \frac{X_{i-1}(l_{i-1})}{K(l_{i-1})}\left[M_{i-1}\omega^2 - K_{i-1}\right]\bar{S}_2(x - l_{i-1})H(x - l_{i-1}) \tag{48}$$

Setting $i = n$, $X_0 = 1$ and using the boundary condition at $x = L$ one obtains the frequency equation as follows

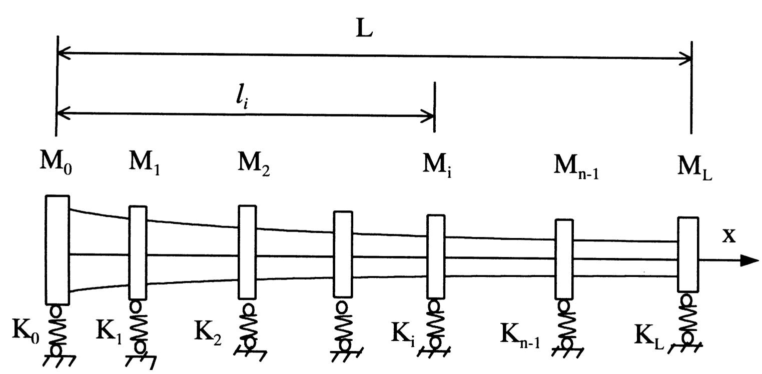

Fig. 3. A one-step beam with spring-spring end and concentrated mass and rectilinear springs at the ends and the other $(n-1)$ points.

$$K(L)X'_{n-1}(L) - \frac{K(L)}{K(l_{n-1})}\left[M_{n-1}\omega^2 - K_{i-1}\right]X_{n-1}(l_{n-1})\bar{S}'_2(L - l_{n-1}) = X_n(L)\left(M_L\omega^2 - K_L\right) \tag{49}$$

or

$$K(L)\left\{\bar{S}'_1(L) - \frac{S'_2(L)}{K(0)}\left(M_0\omega^2 - K_0\right) - \sum_{i=1}^{n}\frac{X_i(l_i)}{K(l_i)}\left[M_i\omega^2 - K_i\right]\bar{S}'_2(L - l_i)\right\}$$
$$= \left(M_L\omega^2 - K_L\right)\left\{\bar{S}_1(L) - \frac{\bar{S}_2(L)}{K(0)}\left(M_0\omega^2 - K_0\right) - \sum_{i=1}^{n}\frac{X(l_i)}{K(l_i)}\left[M_i\omega^2 - K_i\right]\bar{S}_2(L - l_i)H(L - l_i)\right\} \tag{50}$$

in which $M_n = M_L$

7. *Fixed-fixed beam* with concentrated masses and rectilinear springs at the $(n-1)$ intermediate points.

   In this case, the mode shape function of the first segment has the same form as that expressed in Eq. (36), and $X_i(x)$ has the same form as described in Eq. (38), the frequency equation is

$$X_1(L) = \sum_{i=1}^{n-1}\frac{X_i(l_i)}{K(l_i)}\left[M_i\omega^2 - K_i\right]\bar{S}_2(L - l_i) \tag{51}$$

3. Solutions for multi-step beams

A multi-step beam consists of n segments, each segment is a non-uniform shear beam, with concentrated masses and rectilinear springs, as shown in Fig. 4.

If we use $\bar{S}_{i1}$ and $\bar{S}_{i2}$ to represent the basic solutions of the ith step beam, then the mode shape function of the ith step beam can be expressed as

$$X_i(x) = X_i(0)\bar{S}_{i1}(x) + \frac{Q_i(0)}{K_i(0)}\bar{S}_{i2}(x) \tag{52}$$

Since the displacement of the right end of the $(i-1)$th segment is equal to that of left end of the ith segment, one yields

$$X_i(0) = X_{i-1}(l_{i-1}) \tag{53}$$

While the shear force has a jump, i.e.

$$Q_i(0) = Q_{i-1}(l_{i-1}) - \left(M_{i-1}\omega^2 - K_{i-1}\right)X_{i-1}(l_{i-1}) \tag{54}$$

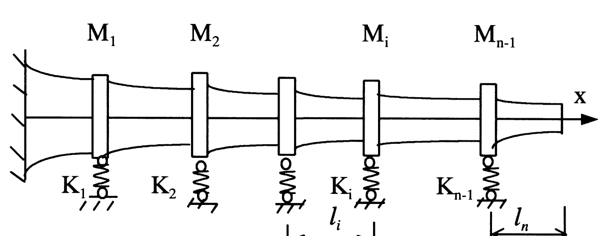

Fig. 4. A fixed-free multi-step beam.

or

$$Q_i(0) = K_{i-1}(l_{i-1})X'_{i-1}(l_{i-1}) - \left(M_{i-1}\omega^2 - K_{i-1}\right)X_{i-1}(l_{i-1}) \tag{55}$$

the mode shape function of the ith segment can be expressed as

$$X_i(x) = X_{i-1}(l_{i-1})\bar{S}_{i1}(x) + \frac{K_{i-1}(l_{i-1})}{K_i(0)}\left[X'_{i-1}(l_{i-1}) - \frac{X_{i-1}(l_{i-1})}{K_{i-1}(l_{i-1})}\left(M_{i-1}\omega^2 - K_{i-1}\right)\right]\bar{S}_{i2}(x) \tag{56}$$

in which l_{i-1} is the length of the $(i-1)$th segment, the origin of the co-ordinate is selected at the left end of this segment.

Eq. (54) is a recurrence formula. According to $X_1(x)$ and Eq. (55), we can determine the mode shape functions of other segments.The frequency equation of this kind of multi-step shear beam can be obtained by using the boundary conditions

1. *Fixed-free beam* (Fig. 4). If the left end of the first segment is fixed, and the right end of the last segment is free, then, the mode shape function of the first segment has the same expression as that given in Eq. (36). Setting $Q_0/K(0) = 1$ and using Eq. (55) one obtains the mode shape functions of all the other segments ($i = 2, 3, \ldots, n$). The frequency equation is

$$X'_n(l_n) = 0 \tag{57}$$

Using Eq. (56) and setting $i = n$, we have

$$X_{n-1}(l_{n-1})\bar{S}'_{n1}(l_n) + \frac{K_{n-1}(l_{n-1})}{K_n(0)}\left[X'_{n-1}(l_{n-1}) - \frac{X_{n-1}(l_{i-1})}{K_{n-1}(l_{n-1})}\left(M_{n-1}\omega^2 - K_{n-1}\right)\right]\bar{S}'_{n2}(l_n) = 0 \tag{58}$$

2. *Fixed-fixed beam*. If the left end of the first segment and the right end of the last segment are fixed, then, the mode shape function of the first segment has the same form as that expressed in Eq. (36), but the frequency equation is

$$X_n(l_n) = 0 \tag{59}$$

or

$$X_{n-1}(l_{n-1})\bar{S}_{n1}(l_n) + \frac{K_{n-1}(l_{n-1})}{K_n(0)}\left[X'_{n-1}(l_{n-1}) - \frac{X_{n-1}(l_{i-1})}{K_{n-1}(l_{n-1})}\left(M_{n-1}\omega^2 - K_{n-1}\right)\right]\bar{S}_{n2}(l_n) = 0 \tag{60}$$

3. *Free-free beam*. If the left end of the first segment and the right end of the last segment are free, then the mode shape function of the first segment has the same form as that given in Eq. (39). Setting $X_0 = 1$ and using Eq. (56) we can determine the mode shape functions of all the other segments ($i = 2, 3, \ldots, n$). The frequency equation that can be established by using the boundary condition, $X'_n(l_n) = 0$, at the right end of the last segment is the same as Eq. (58).
4. *Fixed-spring beam with a concentrated mass at the spring end*. If the left end of the first segment is fixed, then the mode shape function has the same form as Eq. (36). Setting $Q_0/K(0) = 1$ and using Eq. (56) one obtains the mode shape functions of all the other segments ($i = 2, 3, \ldots, n$). The frequency equation can be established by use of the boundary condition, described in Eq. (41), at $x = l_n$, the right end of the last segment as

$$K_n(l_n)X'_n(l_n) = X_n(l_n)\left(M_L\omega^2 - K_L\right) \tag{61}$$

or

$$K_n(l_n)\left\{X_{n-1}(l_{n-1})\bar{S}'_{n1}(l_n)+\frac{K_{n-1}(l_{n-1})}{K_n(0)}\left[X'_{n-1}(l_{n-1})-\frac{X_{n-1}(l_{i-1})}{K_{n-1}(l_{n-1})}\left(M_{n-1}\omega^2-K_{n-1}\right)\right]\bar{S}'_{n2}(l_n)\right\}$$
$$=\left(M_L\omega^2-K_L\right)\left\{X_{n-1}(l_{n-1})\bar{S}_{n1}(l_n)+\frac{K_{n-1}(l_{n-1})}{K_n(0)}\left[X'_{n-1}(l_{n-1})-\frac{X_{n-1}(l_{i-1})}{K_{n-1}(l_{n-1})}\left(M_{n-1}\omega^2-K_{n-1}\right)\right]\bar{S}_{n2}(l_n)\right\} \quad (62)$$

5. *Spring-spring beam with concentrated masses at the ends*. If the left end of the first segment is spring support with a concentrated mass, then the mode shape function of the beam has the same form as that given in Eq. (44). Setting $X_0 = 1$ and using Eq. (56), one obtains the mode shape functions of all the other segments ($i = 2, 3, \ldots, n$). The frequency equation can be established by use of the boundary condition at the right end that is supported by a spring and with a concentrated mass in the last segment.

4. Numerical example 1

To illustrate the proposed method, analysis of free vibrations of a 16-story building with 49.0 m height located in Beijing (Li et al., 1994) is considered herein. Based on the field measurement of this building (Li et al., 1994), it can be simplified as a multi-step cantilever shear beam in analysis of free vibration in the horizontal direction.

The procedure for determining the natural frequencies and mode shapes of this building is as follows

1. *Evaluation of the values of mass per unit length and shear stiffness*. The space floor area of each story is 900 m^2. The mass per unit area and the height of the first story are 1490 kg/m^2 and 4 m, respectively. Thus, the mass per unit length in the vertical direction for the first story is found as

$$\bar{m}_1 = \frac{1490 \times 900}{4} = 3.353 \times 10^5 \text{ kg/m}$$

The mass per unit area and the story height for the building stories from the second to the forth story are1120 kg/m^2 and 3 m, respectively. The mass per unit length in the vertical direction from the second to the forth story is determined as

$$\bar{m} = \frac{1120 \times 900}{3} = 3.36 \times 10^5 \text{ kg/m}$$

It can be seen that $\bar{m}_1$ and $\bar{m}$ are almost identical. This means that the mass distribution is approximately uniform from the first to the fourth story. Thus, this part of building (from the first to the fourth story) is treated as a one-step uniform shear beam (the first step in Fig. 5) for free vibration analysis. The mass per unit length in the vertical direction for the first step beam is taken as 3.36×10^5 kg/m.

The whole building is divided into four step shear beams with uniformly distributed mass for free vibration analysis (Fig. 5). The values of mass per unit length for the second, third and fourth step are found as:

$$\bar{m}_2 = 3.18 \times 10^5 \text{ kg/m}, \quad \bar{m}_3 = 3.07 \times 10^5 \text{ kg/m}, \quad \bar{m}_4 = 2.89 \times 10^5 \text{ kg/m}$$

The structural stiffness which is equal to the sum of the frame stiffness and the filler wall stiffness are

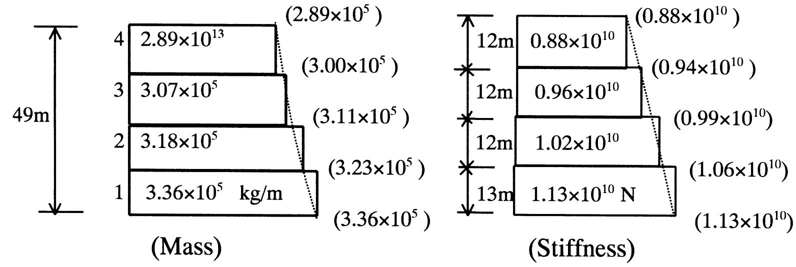

Fig. 5. The distributions of mass and stiffness of a 16-story building.

determined by the formulas given by Li et al. (1994) as follows:

$$K_1 = 1.13 \times 10^{10}\ \mathrm{N}, \quad K_2 = 1.02 \times 10^{10}\ \mathrm{N}, \quad K_3 = 0.96 \times 10^{10}\ \mathrm{N}, \quad K_4 = 0.88 \times 10^{10}\ \mathrm{N}$$

Fig. 5 shows the distributions of mass and stiffness.

2. *Selection of expressions for describing the mass per unit length and shear stiffness*. According to the real variations of mass and stiffness shown in Fig. 5, it is assumed that the distribution of mass and stiffness can be treated as continuously varying distributions described by the following forms

$$K(x) = K_0\, \mathrm{e}^{-\beta \frac{x}{L}} \qquad \bar{m}(x) = \bar{m}_0\, \mathrm{e}^{-\gamma \frac{x}{L}} \tag{63}$$

According to the values of $\bar{m}(x)$ and $K(x)$ at $x = 0$ and $x = L$ one obtains

$$K(0) = K_0 = 1.13 \times 10^{11}\ \mathrm{N}, \qquad \beta = \ln \frac{K(0)}{K(L)} = \ln \frac{1.13}{0.88} = 0.25$$

$$\bar{m}(0) = \bar{m}_0 = 3.36 \times 10^5\ \mathrm{kg/m}, \qquad \gamma = \ln \frac{\bar{m}(0)}{\bar{m}(L)} = \ln \frac{3.36}{2.89} = 0.15$$

The distributions of mass and stiffness given by Eq. (63) are also shown in Fig. 5 (dotted line and values in parentheses) for comparison purposes.

3. *Determination of* $P(r)$. According to Eqs. (63) and (2) one yields

$$p(r) = ae^{br}, \quad r = \int K^{-1}(x)\, \mathrm{d}x = \frac{L}{K_0 \beta}\, \mathrm{e}^{\beta \frac{x}{L}} \tag{64}$$

in which

$$a = \bar{m}_0 K_0, \qquad b = -(\beta + \gamma) \tag{65}$$

4. *Determination of the natural frequencies and mode shape functions*. Since $P(r)$ is a special case of the Case 1, the general solution for mode shape function can be thus found from Eq. (11) and expressed as

$$X(x) = C_1 J_0\left(\alpha\, \mathrm{e}^{br/2}\right) + C_2 Y_0\left(\alpha\, \mathrm{e}^{br/2}\right) \tag{66}$$

Using Eqs. (33) and (66) one obtains the basic solutions as

$$\bar{S}_1(x) = \frac{1}{S_1(0)S_2'(0) - S_2(0)S_1'(0)}\left[S_2'(0)S_1(x) - S_1'(0)S_2(x)\right]$$

$$\bar{S}_2(x) = \frac{1}{S_1(0)S_2'(0) - S_2(0)S_1'(0)}\left[-S_2(0)S_1(x) + S_1(0)S_2(x)\right] \tag{67}$$

in which

$$S_1(x) = J_0\left(\alpha\, e^{br/2}\right), \qquad S_2(x) = Y_0\left(\alpha\, e^{br/2}\right)$$

$$S_i'(0) = \frac{dS_i(x)}{dx}\Big|_{x=0}, \quad i = 1, 2$$

$$\frac{dS_1(x)}{dx} = -J_1\left(\alpha\, e^{\frac{bL}{2K_0\beta}\, e^{\beta\{x/L\}}}\right)\frac{\alpha b}{2K_0}\, e^{\left(\frac{bL}{2K_0\beta}\, e^{\beta\{x/L\}} + \beta\frac{x}{L}\right)}$$

$$\frac{dS_2(x)}{dx} = -Y_1\left(\alpha\, e^{\frac{bL}{2K_0\beta}\, e^{\beta\{x/L\}}}\right)\frac{\alpha b}{2K_0}\, e^{\left(\frac{bL}{2K_0\beta}\, e^{\beta\{x/L\}} + \beta\frac{x}{L}\right)}$$

$$\alpha = \frac{2\omega(\bar{m}_0K_0)^{1/2}}{\beta + \gamma} \tag{68}$$

According to the basic solutions and the boundary conditions at the base, we have

$$X(x) = \frac{Q_0}{K(0)}\bar{S}_2(x) \tag{69}$$

Setting$Q_0/K(0) = 1$ and using the boundary condition at the top, obtain the frequency equation as follows

$$J_0\left(\alpha\, e^{\frac{bL}{2K_0\beta}}\right)Y_1\left(\alpha\, e^{\frac{bL}{2K_0\beta}}\, e^{\beta}\right) = Y_0\left(\alpha\, e^{\frac{bL}{2K_0\beta}}\right)J_1\left(\alpha\, e^{\frac{bL}{2K_0\beta}}\, e^{\beta}\right) \tag{70}$$

Solving Eq. (70) obtains a set of α_i ($i = 1, 2, \ldots$), the minimum value of α_i is found as $\alpha_1 = 1.8112 \times 10^9$.

Using Eq. (68) one obtains that

$$\omega_1 = 5.8789 \text{ rad/s}, \qquad T_1 = 1.0688 \text{ s}.$$

The field measured value of the fundamental natural period is 1.05 s (Li et al., 1994). It is evident that the computed value of the fundamental natural period is very close to the measured one.

As is well known, several simple formulas for calculating the fundamental natural frequency of vibration of buildings have been proposed. For example, The Australia Standard for Earthquake Loading (AS 1170.4) gave

$$T_1 = \frac{H}{46} \tag{71}$$

where H is the building height (m).

Table 1
The fundamental mode shape

x/L	0	0. 125	0. 25	0. 375	0. 50	0. 625	0. 75	0. 875	1. 0
Calculated values of $X_1(x/L)$	0	0. 1897	0. 3764	0. 5479	0. 6988	0. 8293	0. 9107	0. 9679	1.0
Measured values of $X_1(x/L)$	0	0. 185	0. 374	0. 548	0. 701	0. 831	0. 911	0. 971	1.0

Using Eq. (71) one obtains $T_1 = 1.0652$ s for this building.

Based on series of full scale measurements of structural dynamic characteristics of tall buildings, Li et al. (1994) proposed the following empirical formula for calculating the fundamental natural frequency of vibration of buildings:

$$T_1 = 0.072Z \tag{72}$$

where Z is the number of building stories.

Using Eq. (72) one obtains $T_1 = 1.152$ s for this building.

It is clear that the results determined by the empirical formulas (71) and (72) are close to the value calculated by the proposed method.

Substituting α_1 into (58) obtains the fundamental mode shape, $X_1(x)$, the values of which are listed in Table 1.

The field measured values of the fundamental mode shape (Li et al., 1994) are also listed in Table 1. It can be seen from Table 1 that the calculated values of the fundamental mode shape show good agreement with the measured data.

Using the aforementioned procedure, the natural frequencies and mode shapes of higher modes can be determined. The natural frequencies of the second, third mode shapes are found as $\omega_2 = 16.9763$ rad/s, $\omega_3 = 27.8945$ rad/s, and the corresponding mode shapes are presented in Table 2.

5. Numerical example 2

This numerical example illustrates how to determine the natural frequencies and mode shapes of a one-story industrial building in terms of the procedure proposed in this paper. A transverse frame of the building is shown in Fig. 6a.

Because the total weight of all the columns is much less than that of the roof system, all the columns are simplified as weightless. The roof system is idealized as a uniform shear beam with spring supports representing closely spaced columns shown in Fig. 6b. The lateral stiffness of the left and right end walls is much greater than that of a transverse frame, they are thus treated as hinged supports as shown in Fig. 6b. (Wang, 1978; Cao et al., 1992)

Table 2
The second and third mode shapes

x/L	0	0. 125	0. 25	0. 375	0. 50	0. 625	0. 875	1.0
$X_2(x/L)$	0	0. 5793	1. 0791	1. 0842	0. 8929	0. 7095	− 0. 5637	− 1.0
$X_3(x/L)$	0	0. 8907	1. 2617	0. 6074	− 0.3914	−1. 2573	− 0. 1379	1.0

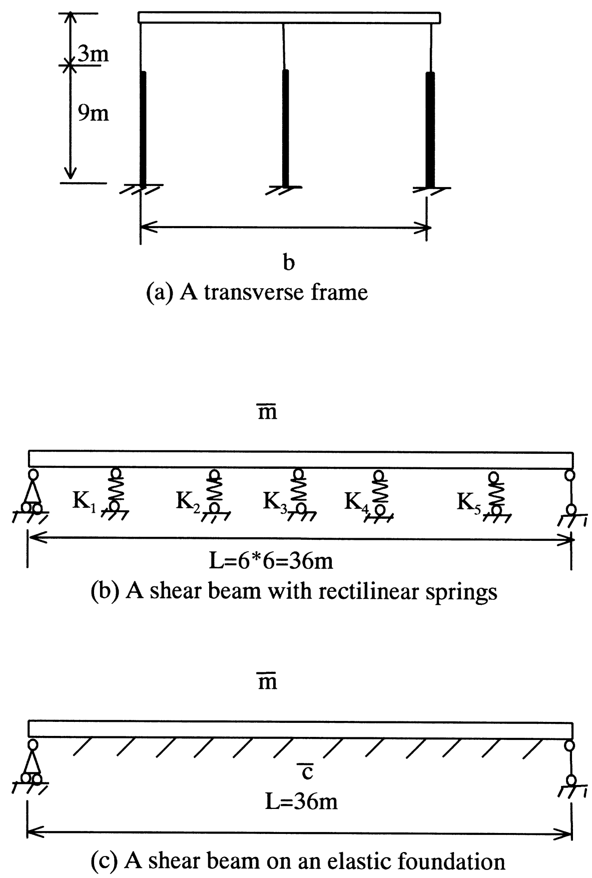

Fig. 6. A one-story industrial building.

The procedure for determining the natural frequencies and mode shapes of this building is as follows

1. *Determination of the stiffness of the springs, the mass per unit length and stiffness of the shear beam is shown in* Fig. 6b. The stiffness of the uniform shear beam representing the roof system is estimated by the following formula proposed by Wang (1978):

$$K = \frac{K_i d b^2}{18.7726 n^2} \tag{73}$$

where K_i is the lateral stiffness of a transverse frame shown in Fig. 6a, which is found as:

$$K_i = 1.71 \times 10^7 \text{ N/m}, \quad i = 1, 2, 3, 4, 5$$

where n is the number of columns of the transverse frame, $n = 6$, d is the lateral length of columns, $d = 6$ m, b is the length of the transverse frame, $b = 36$ m.
Substituting the values of n, d, b and K_i, one obtains

$$K = 7.998 \times 10^8 \text{ N}$$

The mass per unit length of the roof system is found as $\bar{m} = 1.75 \times 10^5$ kg/m

2. *Determination of the basic solutions*. The basic solutions can be determined by using Eqs.(12) and (32) as follows

$$\bar{S}_1(x) = \cos\left(\sqrt{\frac{\bar{m}}{K}}\omega x\right), \qquad \bar{S}_2(x) = \frac{1}{\sqrt{\frac{\bar{m}}{K}}\omega}\sin\left(\sqrt{\frac{\bar{m}}{K}}\omega x\right) \tag{74}$$

Obviously, the values of $\bar{S}_1(x)$ and $\bar{S}_2(x)$ and their first derivatives at $x = 0$ form a unit matrix as given in Eq. (32).

3. *Determination of the natural frequencies and mode shapes*. The mode shape function of the first segment [0, 6 m] is

$$X_1(x) = \sin\left(\sqrt{\frac{\bar{m}}{K}}\omega x\right) \tag{75}$$

The frequency equation is a special case of Eq. (50), i.e.,

$$\sin\left(\sqrt{\frac{\bar{m}}{K}}\omega L\right) + \sum_{i=1}^{n-1}\frac{K_i X_i(l_i)}{\sqrt{K\bar{m}}\omega}\sin\left[\sqrt{\frac{\bar{m}}{K}}\omega(L - l_i)\right] = 0 \tag{76}$$

$X_i(x)$ can be determined by using the following recurrence formula which can be found from Eq. (48) as follows

$$X_i(x) = X_{i-1}(x) + \frac{K_{i-1}}{K}X_{i-1}(l_{i-1})\sin\left[\sqrt{\frac{\bar{m}}{K}}\omega(x - l_{i-1})\right]H(x - l_{i-1}) \tag{77}$$

Solving the frequency equation (77) obtains a set of ω_i ($i = 1, 2, \ldots$), the fundamental circular natural frequency, ω_1, is found as

$$\omega_1 = 7.1449 \text{ rad/s}, \qquad T_1 = 0.8794\ s.$$

The field measured value of T_1 is 0.883 s (Cao et al., 1992).

It is evident that the computed value of the fundamental natural period is in good agreement with the field measured one, suggesting that the proposed methods are applicable to engineering application and practice.

Substituting ω_1 into Eq. (75) obtains the fundamental mode shape of the first segment [0, 6 m], then using $X_1(x)$ and Eq. (77) obtains the mode shapes of the other $(n-1)$ segments. The values of the fundamental mode shape of critical sections are calculated and listed in Table 3.

The field measured data of the fundamental mode shape are also listed in Table 3 for comparison purposes. It can be seen from Table 3 that the calculated fundamental mode shape is very close to the field measured one.

If all the rectilinear springs are treated as an elastic foundation (Fig. 6c) and the coefficient of which is determined by

$$\bar{C} = \frac{K_i}{d_i} = \frac{1.71}{6} \times 10^7 = 2.85 \times 10^6 \text{ N/m}$$

Table 3
The fundamental mode shape

x (m)	0	6	12	18	24	30	36
Calculated values	0	0.4992	0.8657	1.0000	0.8657	0.4992	0.0
	(0) [a]	(0.5000)	(0.8660)	(1.0000)	(0.8660)	(0.5000)	(0.0)
Measured values	0	0.494	0.863	1.000	0.863	0.494	0.0

[a] The values in parentheses are calculated based on the model of a beam on an elastic foundation.

then the governing equation of the mode shape function is as

$$K\frac{d^2X}{dx^2} - (\bar{m}\omega^2 - \bar{C})X = 0 \tag{78}$$

The general solution of Eq. (78) is

$$X(x) = X_0 \cos ax + \frac{Q_0}{aK} \sin ax \tag{79}$$

where

$$a^2 = \frac{\bar{m}\omega^2 - \bar{C}}{K} \tag{80}$$

Using the boundary condition at the left end of the beam obtains $X_0 = 0$, and according to the boundary condition at the right end one yields the frequency equation as

$$\sin aL = 0 \tag{81}$$

Thus

$$a_j = \frac{j\pi}{L} \quad i = 1, \ldots \tag{82}$$

Substituting Eq. (80) into (82) one obtains

$$\omega_j = \sqrt{\left(\frac{j\pi}{L}\right)^2 \frac{K}{\bar{m}} + \frac{\bar{C}}{\bar{m}}} \tag{83}$$

Setting $j = 1$ gives that

$$\omega_1 = 7.1484 \text{ rad/s}, \qquad T_1 = 0.8790 \text{ s}$$

Substituting ω_1 into Eq. (79) and setting $X_0 = 0$, $Q_0/(aK) = 1$, obtain the fundamental mode shape which is also listed in Table 3.

It is shown from the calculated results presented above that the discrete rectilinear springs can be treated as an elastic foundation.

6. Conclusions

A new exact method that combines the basic solutions with unit matrix property and recurrence formula developed, is presented herein to determine the natural frequencies and mode shapes of non-uniform shear beams with classical or non-classical boundary conditions. The function for describing the distribution of mass is arbitrary, and the distribution of shear stiffness is expressed as a functional relation with the mass distribution and vice versa. The governing equation for free vibration of a non-uniform shear beam is reduced to a differential equation of the second-order without the first-order derivative by means of functional transformation. Then, this kind of differential equation is reduced to Bessel equations and other solvable equations for six cases. The exact solutions of mode shape functions are thus found. Since the relation between the mass distribution and stiffness distribution is a functional expression, an exact solution derived represents a class of exact solutions. The basic solutions, which have a unit matrix property, are derived and used to obtain the frequency equations and mode shapes of multi-step shear beams with varying cross-sections. Numerical examples demonstrate that the calculated natural frequencies and mode shapes of shear-type symmetric buildings are very close to the corresponding full-scale measurements, suggesting that the proposed methods are applicable to engineering application and practice. It is also shown that a shear beam with support of the discrete rectilinear springs can be treated as the beam on an elastic foundation for free vibration analysis.

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